1. For the following utility functions,
   - find the marginal rate of substitution,
   - discuss how $MRS_{XY}$ changes as the consumer substitutes $X$ for $Y$ along an indifference curve,
   - derive the equation for the indifference curve where utility is equal to a value of 100, and
   - graph the indifference curve where utility is equal to a value of 100:
     a. $U(X,Y) = XY^2$
     b. $U(X,Y) = XY + 10Y$
     c. $U(X,Y) = X^2 + Y^2$

   **Solution**

   a.
   
   
   $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{Y^2}{2XY} = \frac{Y}{2X}$
   
   To determine how $MRS$ changes as $X$ increases, we can take the partial derivative with respect to $X$:
   
   \[
   \frac{\partial MRS_{XY}}{\partial X} = \frac{\partial (\frac{Y}{2X})}{\partial X} = \frac{\partial (0.5XY^{-1})}{\partial X} = -0.5YX^{-2} < 0
   \]
   
   Therefore, $MRS$ decreases as the consumer increases consumption of $X$ along an indifference curve so this indifference curve is convex to the origin. For $U = 100$,
   
   \[
   U(X,Y) = U = 100 = XY^2
   \]
   
   $Y^2 = 100X^{-1}$
   
   $Y = 10X^{-0.5}$
   
   ![Graph](graph1.png)

   b.
   
   
   $MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{Y}{X + 10}$
   
   To determine how $MRS$ changes as $X$ increases, we can take the partial derivative with respect to $X$:
   
   \[
   \frac{\partial MRS_{XY}}{\partial X} = \frac{\partial (\frac{Y}{X + 10})}{\partial X} = \frac{\partial ((X + 10)^{-1})}{\partial X} = -(X + 10)^{-2} < 0
   \]
   
   Therefore, $MRS$ decreases as the consumer increases consumption of $X$ along an indifference curve so this indifference curve is convex to the origin. For $U = 100$,
   
   \[
   U(X,Y) = U = 100 = XY + 10Y = Y(X + 10)
   \]
   
   $Y = \frac{100}{X + 10}$
   
   Note that the consumer gets positive utility if $X = 0$ so the indifference curve will intersect the vertical axis. For $U = 100$, $Y = 10$ when $X = 0$.  
   
   ![Graph](graph2.png)
c. 

\[ MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{2X}{2Y} = \frac{X}{Y} \]

To determine how \( MRS \) changes as \( X \) increases, we can take the partial derivative with respect to \( X \):

\[ \frac{\partial MRS_{XY}}{\partial X} = \frac{\partial \left( \frac{X}{Y} \right)}{\partial X} = \frac{1}{Y} > 0 \]

for any positive value of \( Y \).

Therefore, \( MRS \) increases as the consumer increases consumption of \( X \) along an indifference curve so this indifference curve is concave to the origin and violates an assumption about consumer preferences: The more a consumer has of a particular good, the less she is willing to give up of something else to get even more of that good.

For \( U = 100 \),

\[ U(X,Y) = \overline{U} = 100 = X^2 + Y^2 \]

\[ Y^2 = 100 - X^2 \]

\[ Y = \sqrt{100 - X^2} \]

2. Suppose that Renée consumes 2 goods, \( X \) and \( Y \), and that her preferences are represented by the utility function \( U(X,Y) = X^{0.5}Y^{0.5} \). Determine whether each of the following utility functions also represent Renée’s preferences:

a. \( V(X,Y) = 33XY + 427 \)

b. \( W(X,Y) = 400X^{0.75}Y^{0.25} \)

2. We know that a utility function is a monotonic transformation of another utility function if the marginal rates of substitution do not change. First, calculate the marginal rate of substitution for Renée’s utility function:

\[ MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{0.5X^{-0.5}Y^{0.5}}{0.5X^{0.5}Y^{-0.5}} = \frac{Y}{X} \]

a. Find the marginal rate of substitution for \( V(X,Y) = 33XY + 427 \):

\[ MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{33Y}{33X} = \frac{Y}{X} \]

This utility function has the same \( MRS \) as the original so it is a monotonic transformation, and therefore, also represents Renée’s preferences.

b. Find the marginal rate of substitution for \( W(X,Y) = 400X^{0.75}Y^{0.25} \):

\[ MRS_{XY} = \frac{MU_X}{MU_Y} = \frac{(0.75)400X^{-0.25}Y^{0.25}}{(0.25)400X^{0.75}Y^{-0.75}} = \frac{3Y}{X} \]

This utility function does not have the same \( MRS \) as the original so it does not represent the same set of preferences as the original utility function.
3. Suppose that a consumer has a choice between gallon jugs of milk and half-gallon jugs of milk. The consumer only cares about how much milk she consumes and does not care about the size of the jug.

a. Write a utility function for this consumer.

b. Suppose the price of a half-gallon jug of milk is $2, the price of a gallon jug of milk is $5, and the consumer has $10 to spend on milk. Set up a Lagrangian and derive the first-order conditions for the maximization problem.

c. Is there a solution to the first-order conditions? If so, solve for the optimal quantities of gallon jugs and half-gallon jugs. If not, what is the solution to the maximization problem?

3. a. The utility function for perfect substitutes is often tricky. If the consumer only cares about how much milk he gets, then he will always trade a gallon jug for two half-gallon jugs. In utility terms, the marginal utility of a gallon jug will always be twice the marginal utility of a half-gallon jug. If $G$ represents gallon jugs and $H$ represents half-gallon jugs, the following utility function—or any monotonic transformation of this utility function—will represent the consumer’s utility:

$$U(G,H) = 2G + H$$

b. The Lagrangian is

$$\max_{G,H,\lambda} L = 2G + H + \lambda(10 - 5G - 2H)$$

FOC:

$$\frac{\partial L}{\partial G} = 2 - 5\lambda = 0$$

$$\frac{\partial L}{\partial H} = 1 - 2\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 10 - 5G - 2H = 0$$

c. The first-order conditions require that $\lambda = \frac{2}{5}$ and $\lambda = \frac{1}{2}$. Therefore, the system of equations has no solution. Utility must be maximized at a consumption bundle where the consumer buys all gallon jugs or all half-gallon jugs. In other words, there is a corner solution. Note that

$$\frac{MU_G}{P_G} = \frac{2}{5} < \frac{MU_H}{P_H} = \frac{1}{2}$$

So, the additional utility per dollar is higher for half-gallon jugs; thus, the consumer will want to buy only half-gallon jugs. Alternatively, you could solve this graphically by choosing the highest possible indifference curve.

4. Terry has $100 to spend on lake perch and walleye for his family’s Friday fish fry. Terry’s utility function is

$$U(P,W) = 10W + PW$$

where $P$ is pounds of lake perch and $W$ is pounds of walleye.

a. Suppose the price of perch is $5 per pound and the price of walleye is $3 per pound. Find Terry’s optimal combination of perch and walleye.

b. Suppose that the price of perch increases to $12 per pound, while the price of walleye remains unchanged. Find Terry’s optimal combination of perch and walleye.

4. a. Write out the maximization problem and the Lagrangian:

$$\max_{P,W} 10W + PW \text{ s.t. } 100 = 5P + 3W$$

$$\max_{P,W,\lambda} L = 10W + PW + \lambda(100 - 5P - 3W)$$

FOC:

$$\frac{\partial L}{\partial P} = W - 5\lambda = 0$$

$$\frac{\partial L}{\partial W} = 10 + P - 3\lambda = 0$$

$$\frac{\partial L}{\partial \lambda} = 100 - 5P - 3W = 0$$
From the first two conditions,
\[ \lambda = \frac{W}{12} = \frac{10 + P}{3} \]
\[ 3W = 120 + 12P \]
Substituting into the third FOC:
\[ 100 - 12P - 3W = 100 - 12P - (120 + 12P) = 0 \]
\[ P = \frac{5}{6} \]
\[ P \text{ cannot be negative; instead, Terry will consume no perch. Therefore, this is a corner solution where} \]
\[ P = 0. \text{ From the budget constraint,} \]
\[ W = \frac{100}{3} = 33\frac{1}{3}. \]

b. Write out the maximization problem and the Lagrangian:
\[ \max_{P,W} 10W + PW \text{ s.t. } 100 = 12P + 3W \]
\[ \max_{P,W,\lambda} C = 10W + PW + \lambda(100 - 12P - 3W) \]
FOC:
\[ \frac{\partial C}{\partial P} = W - 12\lambda = 0 \]
\[ \frac{\partial C}{\partial W} = 10 + P - 3\lambda = 0 \]
\[ \frac{\partial C}{\partial \lambda} = 100 - 12P - 3W = 0 \]
From the first two conditions,
\[ \lambda = \frac{W}{12} = \frac{10 + P}{3} \]
\[ 3W = 120 + 12P \]
Substituting into the third FOC:
\[ 100 - 12P - 3W = 100 - 12P - (120 + 12P) = 0 \]
\[ P = \frac{5}{6} \]